AIAA JOURNAL
Vol. 33, No. 10, October 1995

Free Vibration of Delaminated Composite Sandwich Beams

Jian Shiun Hu*
National Center for High-Performance Computing, HsinChu, Taiwan, Republic of China
and
Chyanbin Hwu†
National Cheng Kung University, Tainan, Taiwan, Republic of China

Consider a sandwich plate with anisotropic composite laminated faces and an ideally orthotropic honeycomb core. In this paper, a one-dimensional model considering the transverse shear effect and rotary inertia for the free vibration analysis of a sandwich plate with an across-the-width delamination located at the interface between the upper face and core is developed. With this model, the natural frequencies and mode shapes of the delaminated composite sandwich beams can be obtained by solving the eigenvalues and eigenvectors of 12 simultaneous homogeneous algebraic equations. Because there are no such general solutions presented in the literature, verification is done by some special cases such as delaminated composite beams (without core) and perfect composite sandwich beams (without delamination). Based upon this general solution, the effects of faces, core, and delamination on free vibration behavior of composite sandwiches are studied thoroughly.

I. Introduction

THERE are many advantages of composite sandwiches over the conventional structural materials, such as high bending stiffness, low specific weight, and good thermal and acoustical insulation. However, these new materials also induce some new problems. One of them is delamination, which may occur either on the interply of composite laminated faces or on the interface between face and core. It is, therefore, important to know the effect of delamination on some mechanical problems like free vibration.

To study the delamination effect on free vibration, Kulkarni and Frederick[1] considered a circular cylindrical shell with a circumferentially symmetric delamination of small length at the middle surface and proposed that the natural frequency was a parameter that reflects the debonding or weakening of the composite. Ramkumar et al.[2] studied the free vibration of composite beams with through-width delamination and compared the theoretical results with the results of vibration experiments on a debonded laminated cantilever beam. The analytical prediction found to be consistently much lower than the experimental values indicates that the residual bending stiffness was grossly underestimated in their analysis. Cawley and Adams[3] reported that the shift in natural frequency due to the delamination provides the basis for nondestructive testing via vibration technique. By considering the coupling of longitudinal and flexural motions in the split region, Wang et al.[4] developed a theoretical model to analyze the free vibration of split isotropic beams. Later, Mujumdar and Suryanarayan[5] studied the effect of delamination on flexural vibration of an isotropic beam under the assumption that the split regions of the beam should be constrained to move together in the transverse direction. Tracy and Pardoen[6] studied the effect of delamination on natural frequency of symmetric laminated beam containing midplane delamination by using the Euler beam theory and verified the results by experiments and the finite element method. Recently, Shen and Grady[7] used the Galerkin method to analyze the delamination effect on natural frequency and vibration mode shape of composite laminated beam and verified their results by experiments, in which the couplings between longitudinal vibration and transverse bending motion were also considered.

The flexural vibration analyses of perfect sandwich plates were presented earlier by Yu,[8–10] in which the sandwich plate was confined to one with isotropic faces and an orthotropic core. Recently, bending and free vibration analyses of sandwich plates with unbalanced anisotropic laminated faces and an orthotropic core were presented by Monforton and Ibrahim,[11,12] Ibrahim et al.,[13] and Kanematsu and Hirano[14] by using the finite element method and the Rayleigh–Ritz method.

As stated in the last two paragraphs, there are many works concerning the vibration of delaminated composites and perfect sandwiches. However, it still lacks research concerning the vibration of delaminated composite sandwiches. In this paper, a one-dimensional model of delaminated composite sandwiches proposed by Hwu and Hu[15] for buckling and postbuckling problems is modified to study free vibration problems of delaminated composite sandwiches. The results are general enough and can be reduced to the cases of delaminated composites or perfect sandwiches, which are used to verify our solutions. Meanwhile, the various effects such as transverse shear modulus and thickness of core, delamination length and location, fiber direction and stacking sequence of laminated faces on natural frequencies, and the associate mode shapes are also studied.

II. Vibration Analysis

Recently, a one-dimensional mathematical model was developed by Hwu and Hu[15] for the buckling and postbuckling of delaminated composite sandwich beams. In that model, the delaminated composite sandwich beam is separated into four regions as shown in Fig. 1. Regions 1 and 4 are considered to be composite sandwich beams with the faces resisting in-plane force N_x and bending moment M_x, and the core undergoing the transverse shear force Q_x, whereas regions 2 and 3 are considered to be special cases of sandwiches that also carry in-plane force, bending moment, and transverse shear force. In all regions, the deformation of the core and faces are assumed to have the form of the Timoshenko beam, i.e.,

$$u = u_0 + z\left(\gamma_{xz} - \frac{\partial w}{\partial x}\right) \tag{1}$$

where u and w denote the displacements in the x and z directions, respectively; u_0 is the midplane axial displacement; and γ_{xz} is the transverse shear strain. Although u is a linear function of z, u_0, γ_{xz}, and w are independent of z and are functions of x only. With this assumption for the sandwich beam deformation, the equations of

Received Aug. 12, 1994; revision received Feb. 10, 1995; accepted for publication March 15, 1995.

*Associate Research Scientist, Research and Promotion Division.

†Professor, Institute of Aeronautics and Astronautics.

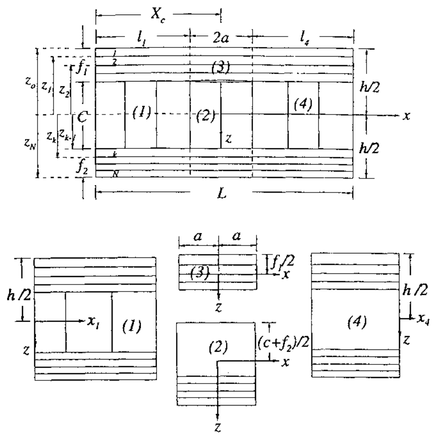

Fig. 1 Geometry and notation of delaminated composite sandwich beam.

motion can be derived by a way similar to that described in the paper by Hwu and Hu.[15] The results are

$$\frac{\partial N_x}{\partial x} = 0$$
$$\frac{\partial Q_x}{\partial x} + N_x\frac{\partial^2 w}{\partial x^2} + q = \rho h\frac{\partial^2 w}{\partial t^2} \tag{2a}$$
$$\frac{\partial M_x}{\partial x} = Q_x + I\frac{\partial^2 \beta_x}{\partial t^2}$$

where

$$I = \frac{\rho h^3}{12}, \qquad \beta_x = \gamma_{xz} - \frac{\partial w}{\partial x} \tag{2b}$$

and I, ρ, h, and β_x are, respectively, the moment of inertia (with respect to the midplane), mass density, thickness, and rotation angle of the sandwich beams; q is the normal pressure distributed over the top or bottom surface of the sandwich beams. The resultant forces N_x, M_x, and Q_x are related to the midplane axial strain ϵ_{x_0}, curvature κ_x, and the transverse shear strain γ_{xz} by

$$N_x = A_{11}\epsilon_{x_0} + B_{11}\kappa_x$$
$$M_x = B_{11}\epsilon_{x_0} + D_{11}\kappa_x \tag{2c}$$
$$Q_x = S\gamma_{xz}$$

where

$$\epsilon_{x_0} = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad \kappa_x = \frac{\partial \beta_x}{\partial x} = \frac{\partial}{\partial x}\left(\gamma_{xz} - \frac{\partial w}{\partial x}\right) \tag{2d}$$

and where A_{11}, B_{11}, D_{11}, and S are, respectively, the extensional, coupling, bending, and transverse shear stiffnesses of the composite sandwich beam. The stiffnesses A_{11}, B_{11}, and D_{11} are contributed by the faces of the sandwiches, whereas the shear stiffness S is mostly contributed by the core. The formulas for calculating A_{11}, B_{11}, and D_{11} are the same as those given in the classical lamination theory except that the plane $z = 0$ is located on the midsurface of the entire sandwich and not the face.[15] To calculate the transverse shear stiffness S, the shear stress distribution is assumed to be uniform across the core and parabolic across the face.[16] Hence,

$$S = cG_{xz}^c + \frac{5}{4}\sum_{k=1}^{N}(\bar{Q}_{55})_k\left[z_k - z_{k-1} - \frac{4}{3}\left(z_k^3 - z_{k-1}^3\right)\frac{1}{h^2}\right]$$

in which $\bar{Q}_{55} = G_{xz}\cos^2\theta + G_{yz}\sin^2\theta$, where G_{xz}, G_{yz}, and θ are, respectively, the transverse shear moduli and fiber direction of the lamina. The terms c and G_{xz}^c are the thickness and effective transverse shear modulus of the core, N is the number of layers of composite sandwich (excluding the core), and z_k is the coordinate of kth layer as shown in Fig. 1. If only the composite laminate faces are considered, the first term cG_{xz}^c should be deleted since it is contributed by the core. It should also be noted that in Eq. (2a) the longitudinal inertia term $\rho\ddot{u}_0$ has been neglected since it is small for the lower flexural modes of beams.[5]

The first equation of Eq. (2a) shows that N_x is a constant throughout the beam and is now set to be the compressive axial load P, i.e., $N_x = -P$. Knowing that P is a constant, by the first and second equations of Eq. (2c) the midplane axial strain ϵ_{x_0} and the bending moment M_x can be expressed in terms of the curvature κ_x, which are

$$\epsilon_{x_0} = -AP - B\kappa_x$$
$$M_x = -BP + D\kappa_x \tag{3a}$$

where

$$A = \frac{1}{A_{11}}, \qquad B = \frac{B_{11}}{A_{11}}, \qquad D = D_{11} - \frac{B_{11}^2}{A_{11}} \tag{3b}$$

If the axial load P is treated as a known value, the three equations of motion shown in Eq. (2a) may be reduced to only one equation expressed by the transverse displacement w. Use of the second equation of Eq. (2a) and the third equation of Eq. (2c) may provide the relation between γ_{xz} and w. By this relation and the second equation of Eq. (2d), the curvature κ_x can be expressed in terms of w. They are

$$\frac{\partial \gamma_{xz}}{\partial x} = \frac{P}{S}\frac{\partial^2 w}{\partial x^2} + \frac{\rho h}{S}\frac{\partial^2 w}{\partial t^2} - \frac{q}{S}$$
$$\kappa_x = \frac{\partial \beta_x}{\partial x} = -\left(1 - \frac{P}{S}\right)\frac{\partial^2 w}{\partial x^2} + \frac{\rho h}{S}\frac{\partial^2 w}{\partial t^2} - \frac{q}{S} \tag{4}$$

Substituting the third equation of Eq. (2c), and the second equations of Eqs. (3a) and (2d) into the third equation of Eq. (2a) and differentiating both sides of the equation with respect to x, one can write the equation of motion as

$$D\frac{\partial^2 \kappa_x}{\partial x^2} = S\frac{\partial \gamma_{xz}}{\partial x} + I\frac{\partial^2 \kappa_x}{\partial t^2} \tag{5a}$$

With the relations provided in Eq. (4), the equation of motion for the composite sandwich beams can now be expressed by only one parameter w,

$$D\left(1 - \frac{P}{S}\right)\frac{\partial^4 w}{\partial x^4} - \left[I\left(1 - \frac{P}{S}\right) + \frac{\rho h D}{S}\right]\frac{\partial^4 w}{\partial x^2 \partial t^2}$$
$$+ \frac{\rho h I}{S}\frac{\partial^4 w}{\partial t^4} + P\frac{\partial^2 w}{\partial x^2} + \rho h\frac{\partial^2 w}{\partial t^2} = -\frac{D}{S}\frac{\partial^2 q}{\partial x^2} + \frac{I}{S}\frac{\partial^2 q}{\partial t^2} + q \tag{5b}$$

If we now consider the problems of free vibration, $q = P = 0$ for regions 1 and 4, and the equations of motion (5b) can be reduced to

$$D_i\frac{\partial^4 w_i}{\partial x_i^4} - \left[I_i + \frac{\rho_i h_i D_i}{S_i}\right]\frac{\partial^4 w_i}{\partial x_i^2 \partial t^2} + \frac{\rho_i h_i I_i}{S_i}\frac{\partial^4 w_i}{\partial t^4}$$
$$+ \rho_i h_i\frac{\partial^2 w_i}{\partial t^2} = 0, \qquad i = 1, 4 \tag{6}$$

Here the subscript i denotes the region number. As suggested by Mujumdar and Suryanarayan,[5] q and P will not be zero for regions 2 and 3. They showed that for the cases of delaminated isotropic beams the free mode model, though mathematically admissible, is not physically feasible since it gives vibration modes with overlaps of deformation that violate compatibility. In reality, however, the tendency of one of the delaminated layers to overlap on the other will be resisted by the development of a contact pressure distribution between the adjacent layers. Such a pressure distribution would constrain the transverse deformation of these adjacent layers to be identical and thus ensure compatibility. The two segments in the delamination region, though having identical transverse displacements, are assumed to be free to slide over each other in the axial direction except at their ends, which are connected to the integral segments. The contact is assumed to be frictionless and uniform. Thus, for regions 2 and 3, we assume

$$w_2 = w_3, \qquad P_2 = -P_3 = P = \text{const}, \qquad q_2 = -q_3 = q \tag{7a}$$

where the normal pressure q can be represented in harmonic form as

$$q = q_0 \sin \omega t, \qquad q_0 = \text{const} \tag{7b}$$

With the preceding assumptions, Eq. (5b) can be simplified. Moreover, by the fact that the rotary inertia I is higher order in h, which is small even for the sandwich beam, and the in-plane forces are usually far smaller than the transverse shear forces for the flexural vibration problems, the terms $\omega^2 I/S$ and P/S might be far less than unity and may be neglected for the convenience of mathematical manipulation. For the sake of prudence, we will check this assumption after the natural frequencies and the mode shapes are obtained. Now, the equations of motion for regions 2 and 3 can be simplified as

$$\begin{aligned} &D_2 \frac{\partial^4 w_2}{\partial x^4} - \left(I_2 + \frac{\rho_2 h_2 D_2}{S_2}\right) \frac{\partial^4 w_2}{\partial x^2 \partial t^2} + \frac{\rho_2 h_2 I_2}{S_2} \frac{\partial^4 w_2}{\partial t^4} \\ &\quad + P \frac{\partial^2 w_2}{\partial x^2} + \rho_2 h_2 \frac{\partial^2 w_2}{\partial t^2} = q \\ &D_3 \frac{\partial^4 w_2}{\partial x^4} - \left(I_3 + \frac{\rho_3 h_3 D_3}{S_3}\right) \frac{\partial^4 w_2}{\partial x^2 \partial t^2} + \frac{\rho_3 h_3 I_3}{S_3} \frac{\partial^4 w_2}{\partial t^4} \\ &\quad - P \frac{\partial^2 w_2}{\partial x^2} + \rho_3 h_3 \frac{\partial^2 w_2}{\partial t^2} = -q \end{aligned} \tag{8}$$

in which the local coordinates x_2 and x_3 are chosen to be the same as the global coordinate x (see Fig. 1). By adding the preceding two equations, the unknown normal contact pressure q and the axial load P can be eliminated. The result is

$$\begin{aligned} &(D_2 + D_3) \frac{\partial^4 w_2}{\partial x^4} - \left(I_2 + I_3 + \frac{\rho_2 h_2 D_2}{S_2} + \frac{\rho_3 h_3 D_3}{S_3}\right) \frac{\partial^4 w_2}{\partial x^2 \partial t^2} \\ &\quad + \left(\frac{\rho_2 h_2 I_2}{S_2} + \frac{\rho_3 h_3 I_3}{S_3}\right) \frac{\partial^4 w_2}{\partial t^4} + (\rho_2 h_2 + \rho_3 h_3) \frac{\partial^2 w_2}{\partial t^2} = 0 \end{aligned} \tag{9}$$

For harmonic motion,

$$w_i(x_i, t) = W_i(x_i) \sin \omega t, \qquad i = 1, 2, 4 \tag{10}$$

in which ω is the natural frequency and W_i is the mode shape for the ith region. Substituting Eq. (10) into Eqs. (6) and (9), one can obtain the general solutions of the differential equations (6) and (9) as

$$\begin{aligned} W_i(x_i) &= \Gamma_{i1} \cosh \lambda_i x_i + \Gamma_{i2} \sinh \lambda_i x_i + \Gamma_{i3} \cos \mu_i x_i \\ &\quad + \Gamma_{i4} \sin \mu_i x_i, \qquad i = 1, 4, \end{aligned} \tag{11a}$$

$$W_2(x) = \Gamma_{21} \cosh \lambda_2 x + \Gamma_{22} \sinh \lambda_2 x + \Gamma_{23} \cos \mu_2 x + \Gamma_{24} \sin \mu_2 x$$

where

$$\begin{aligned} \lambda_i^2 &= \frac{\omega^2}{2D_i}\left\{ -\hat{I}_i + \sqrt{\hat{I}_i^2 + 4\hat{m}_i D_i}\right\}, \qquad i = 1, 4 \\ \mu_i^2 &= \frac{\omega^2}{2D_i}\left\{ \hat{I}_i + \sqrt{\hat{I}_i^2 + 4\hat{m}_i D_i}\right\}, \qquad i = 1, 4 \\ \lambda_2^2 &= \frac{\omega^2}{2(D_2 + D_3)} \\ &\quad \times \left\{ -(\hat{I}_2 + \hat{I}_3) + \sqrt{(\hat{I}_2 + \hat{I}_3)^2 + 4(\hat{m}_2 + \hat{m}_3)(D_2 + D_3)}\right\} \\ \mu_2^2 &= \frac{\omega^2}{2(D_2 + D_3)} \\ &\quad \times \left\{ (\hat{I}_2 + \hat{I}_3) + \sqrt{(\hat{I}_2 + \hat{I}_3)^2 + 4(\hat{m}_2 + \hat{m}_3)(D_2 + D_3)}\right\} \end{aligned} \tag{11b}$$

and

$$\hat{I}_i = I_i + \frac{\rho_i h_i D_i}{S_i}, \qquad \hat{m}_i = \frac{\rho_i h_i}{\omega^2} - \frac{\rho_i h_i I_i}{S_i} \tag{11c}$$

where the various Γ_{ij} are the unknown coefficients to be determined by the continuity conditions and boundary conditions set for the problem.

III. Continuity and Boundary Conditions

Usually, the continuity conditions in the beam problems include the continuity of the transverse displacements, slopes, bending moments, and shear forces. Since the transverse shear effects are considered in all regions, the slopes of all regions are expressed in the form

$$\frac{\partial w_i}{\partial x_i} - \gamma_{xz_i}, \qquad i = 1, \ldots, 4 \tag{12}$$

of which the transverse shear strains γ_{xz_i} obtained from Eq. (4) are given as

$$\begin{aligned} \gamma_{xz_i} &= \frac{\rho_i h_i}{S_i} \int \frac{\partial^2 w_i}{\partial t^2} \, \mathrm{d}x_i, \qquad i = 1, 4 \\ \gamma_{xz_i} &= \frac{P_i}{S_i} \frac{\partial w_2}{\partial x} + \frac{\rho_i h_i}{S_i} \int \frac{\partial^2 w_2}{\partial t^2} \, \mathrm{d}x - \frac{1}{S_i} \int q_i \, \mathrm{d}x, \qquad i = 2, 3 \end{aligned} \tag{13}$$

Since the face made up of composite laminate is much thinner compared with the core thickness, the slope of region 3 is dominated by the derivative of deflection, $\partial w_2/\partial x$, of which transverse shear strain γ_{xz_3} is of little effect. Moreover, region 2 is assumed to vibrate together with region 3. Therefore, the slope of regions 2 and 3 may now be approximated by $\partial w_2/\partial x$ for the convenience of mathematical manipulation, which should be verified after the natural frequency and mode shape are calculated.

With the preceding approximation, the continuity conditions at the crack tips for the present problem can then be expressed as

$$\begin{aligned} &w_1|_{x_1 = \ell_1} = w_2|_{x=-a}, \qquad \left[\frac{\partial w_1}{\partial x_1} - (\gamma_{xz})_1\right]_{x_1 = \ell_1} = \left.\frac{\partial w_2}{\partial x}\right|_{x=-a} \\ &w_2|_{x=a} = w_4|_{x_4=-\ell_4}, \qquad \left.\frac{\partial w_2}{dx}\right|_{x=a} = \left[\frac{\partial w_4}{dx_4} - (\gamma_{xz})_4\right]_{x_4=-\ell_4} \\ &(M_x)_1|_{x_1=\ell_1} = (M_x)_2|_{x=-a} + (M_x)_3|_{x=-a} + (P/2)h \\ &(M_x)_4|_{x_4=-\ell_4} = (M_x)_2|_{x=a} + (M_x)_3|_{x=a} + (P/2)h \\ &(Q_x)_1|_{x_1=\ell_1} = (Q_x)_2|_{x=-a} + (Q_x)_3|_{x=-a} \\ &(Q_x)_4|_{x_4=-\ell_4} = (Q_x)_2|_{x=a} + (Q_x)_3|_{x=a} \end{aligned} \tag{14}$$

It should be noted that the transverse shear force Q_x in all regions is related to γ_{xz} by the third equation of Eq. (2c). The bending moment M_x and the transverse shear strain γ_{xz} are related to the transverse displacement w by the second equation of Eq. (3a) and Eq. (4). With these relations, it is noted that the continuity conditions shown in Eq. (14) can all be expressed in terms of the transverse displacement w_i except that the bending and shear force continuity have the extra unknown axial load P and normal pressure q in which the axial load P may be found by the satisfaction of compatibility of axial displacements at the tip of delamination. That is,

$$\begin{aligned} \left[(u_0)_2 + \frac{1}{2}(c + f_2)\frac{\partial w_2}{\partial x}\right]_{x=-a} &= \left[(u_0)_3 - \frac{f_1}{2}\frac{\partial w_2}{\partial x}\right]_{x=-a} \\ \left[(u_0)_2 + \frac{1}{2}(c + f_2)\frac{\partial w_2}{\partial x}\right]_{x=a} &= \left[(u_0)_3 - \frac{f_1}{2}\frac{\partial w_2}{\partial x}\right]_{x=a} \end{aligned} \tag{15}$$

The midplane axial displacement u_0 of regions 2 and 3 can also be expressed in terms of w by substituting the first equation of Eq. (2d) into the first equation of Eq. (3a) and integrating with respect to x. The results are

$$(u_0)_i = -A_i P_i x + B_i \frac{\partial w_i}{\partial x} - \frac{1}{2}\int_0^x \left(\frac{\partial w_i}{\partial x}\right)^2 dx + \text{const} \quad (16)$$

Substituting Eq. (16) into Eq. (15) and subtracting the second equation (15) by the first equation (15), we obtain

$$P = \frac{(B_3 - B_2) - h/2}{2a(A_2 + A_3)}\left(\frac{\partial w_2}{\partial x}\bigg|_{x=a} - \frac{\partial w_2}{\partial x}\bigg|_{x=-a}\right) \quad (17)$$

By using the relation given in Eq. (17), the other unknown q may be found by the satisfaction of compatibility of transverse shear strain at the crack tip of delamination. However, this approach may lead to a nonlinear equation of Γ_{ij}, which may cause trouble in mathematical manipulation. To avoid that, we ignore q_i/S_i, $i = 2, 3$, since they may be far smaller than the other terms shown in the second equation of Eq. (13). This assumption will also be checked after the natural frequencies and mode shapes are obtained.

The continuity conditions (14) will now provide 8 linear homogeneous algebraic equations in 12 unknown coefficients Γ_{ij} ($i = 1, 2, 4$; $j = 1, 2, 3, 4$). The remaining four equations come from the boundary conditions for both ends of the sandwich beams. In the following, three different boundary conditions will be studied. They are the following:

1) Simply supported ends:

$$w_1 = 0, \quad (M_x)_1 = 0, \quad \text{on } x_1 = 0$$
$$w_4 = 0, \quad (M_x)_4 = 0, \quad \text{on } x_4 = 0 \quad (18a)$$

2) Clamped-clamped ends:

$$w_1 = 0, \quad \frac{\partial w_1}{\partial x_1} - (\gamma_{xz})_1 = 0, \quad \text{on } x_1 = 0$$
$$w_4 = 0, \quad \frac{\partial w_4}{\partial x_4} - (\gamma_{xz})_4 = 0, \quad \text{on } x_4 = 0 \quad (18b)$$

3) Clamped-free ends:

$$w_1 = 0, \quad \frac{\partial w_1}{\partial x_1} - (\gamma_{xz})_1 = 0, \quad \text{on } x_1 = 0$$
$$(M_x)_4 = 0, \quad (Q_x)_4 = 0, \quad \text{on } x_4 = 0 \quad (18c)$$

In the preceding text, all of the boundary conditions provide four equations. Combining these 4 equations with the 8 continuity equations, one obtains 12 linear homogeneous algebraic equations in 12 unknown coefficients Γ_{ij}. The frequencies and mode shapes can be obtained as the eigenvalues and eigenvectors of this equation set.

IV. Special Cases

Since no published analytical results have been found in the literature for the vibration analysis of delaminated composite sandwich beams, the verification will be done by some degenerate cases such as perfect sandwich beams (without delamination) and delaminated composite beams (without core).

A. Perfect Sandwich Beams (Without Delaminations)

For the special case that the delamination does not exist in the sandwich beam, the problem becomes much simpler than those discussed previously. In this case, there is no need to separate the beam into four regions. All we need to do is substitute the boundary conditions (18) into the general solutions of the first equation of Eq. (11a) for the perfect sandwich beams. The natural frequencies ω and the mode shapes $W_n(x)$ for three different boundary conditions of perfect sandwich beams are then obtained as follows.

1) Simply supported ends:

$$\frac{\rho h I}{SD}\omega^4 - \left[\left(\frac{I}{D} + \frac{\rho h}{S}\right)\left(\frac{n\pi}{L}\right)^2 + \frac{\rho h}{D}\right]\omega^2 + \left(\frac{n\pi}{L}\right)^4 = 0$$
$$W_n(x) = \sin\mu_1 x - \frac{\sin\mu_1 L}{\sinh\lambda_1 L}\sinh\lambda_1 x \quad (19a)$$

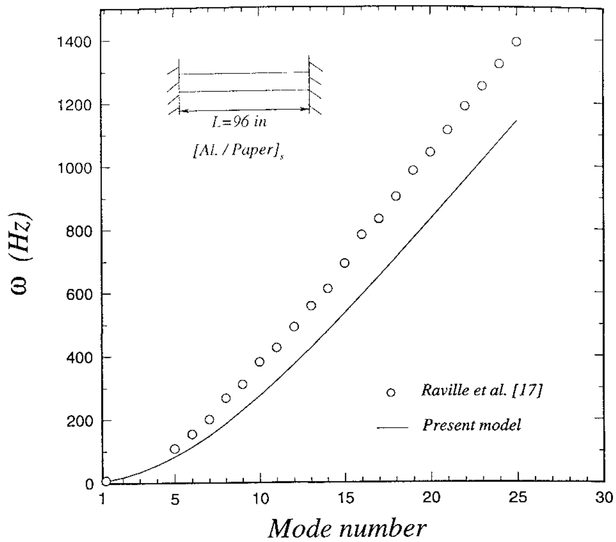

Fig. 2 Comparison of natural frequencies of perfect sandwich beam (beam 5 of Ref. 17).

2) Clamped-clamped ends:

$$2(1 - \cosh\lambda_1 L\cos\mu_1 L) + \left[\Lambda(\omega) - \frac{1}{\Lambda(\omega)}\right] \times \sinh\lambda_1 L\sin\mu_1 L = 0$$
$$W_n(x) = \cosh\lambda_1 x - \cos\mu_1 x + \frac{\cos\mu_1 L - \cosh\lambda_1 L}{\sinh\lambda_1 L - \Lambda(\omega)\sin\mu_1 L} \times (\sinh\lambda_1 x - \Lambda(\omega)\sin\mu_1 x) \quad (19b)$$

3) Clamped-free ends:

$$2\Lambda(\omega) + \frac{\mu_1}{\lambda_1}\left[1 + \frac{\lambda_1^2}{\mu_1^2}\Lambda^2(\omega)\right]\cosh\lambda_1 L\cos\mu_1 L + \left(\frac{\lambda_1}{\mu_1} - \frac{\mu_1}{\lambda_1}\right)\Lambda(\omega)\sinh\lambda_1 L\sin\mu_1 L = 0$$
$$W_n(x) = \cosh\lambda_1 x - \cos\mu_1 x - (\sinh\lambda_1 x - \Lambda(\omega)\sin\mu_1 x) \times \frac{\mu_1\sinh\lambda_1 L - \lambda_1\sin\mu_1 L}{\mu_1\cosh\lambda_1 L + \lambda_1\Lambda(\omega)\cos\mu_1 L} \quad (19c)$$

where

$$\Lambda(\omega) = \frac{\lambda_1 + (\rho h\omega^2/S\lambda_1)}{\lambda_2 - (\rho h\omega^2/S\lambda_2)} \quad (19d)$$

To verify the preceding results, comparison is made with that presented by Raville et al.[17] for the clamped-clamped boundary condition in which the effects of rotary inertia and transverse shear deformation of the face sheets are neglected. Figure 2 shows that the present results are little lower than those given by Raville et al., a which is reasonable because they neglect rotary inertia and transverse shear deformation.

B. Delaminated Composites (Without Core)

An example of delaminated composite beam $[0/90/0/90]_s$ made of T300/934 graphite/epoxy with debonding along interface 2 (given in the paper by Shen and Grady[7]) is studied for the purpose of verification. The results presented in Fig. 3 show that the fundamental resonant frequencies obtained by the present model match well with experimental results as compared with those obtained by the models given by Shen and Grady.[7] These results give us confidence in the present model, which is general and can be reduced to solve the free vibration problems of delaminated composite laminates.

V. Results and Discussion

The geometric notation of delaminated composite sandwich is shown in Fig. 1, in which the face and core of the sandwiches are made of carbon/epoxy and aluminum honeycomb, respectively. The reference material properties of carbon/epoxy are

Table 1 Comparison of P_i/S_i, $\omega_0^2 I_i/S_i$,[a] $\gamma_{xz_i}/(dw_i/dx)$, and $(2aq_i/S_i)/\gamma_{xz_i}$ with unity ([$0_2/90_2/0_2$/honeycomb]$_s$)

$2a/L$	Boundary coditions[b]	P/S_2	P/S_3	$(\gamma_{xz_2}/(dw_2/dx))_{x=0}$	$(\gamma_{xz_3}/(dw_2/dx))_{x=0}$	$((2aq_2/S_2)/\gamma_{xz_2})_{x=0}$	$((2aq_3/S_3)/\gamma_{xz_3})_{x=0}$
0.2	s.s.	1.01×10^{-2}	6.11×10^{-3}	1.25×10^{-3}	6.34×10^{-4}	9.08×10^{-4}	2.22×10^{-4}
	c.c.	5.59×10^{-3}	2.89×10^{-3}	8.73×10^{-3}	4.49×10^{-3}	5.78×10^{-4}	1.3×10^{-3}
	c.f.	1.56×10^{-3}	8.09×10^{-4}	3.21×10^{-3}	1.69×10^{-3}	4.82×10^{-4}	1.12×10^{-3}
0.3	s.s.	8.84×10^{-3}	4.58×10^{-3}	1.94×10^{-3}	9.86×10^{-4}	3.51×10^{-4}	7.68×10^{-4}
	c.c.	3.45×10^{-3}	1.78×10^{-3}	1.57×10^{-2}	7.93×10^{-3}	1.12×10^{-3}	2.49×10^{-3}
	c.f.	7.52×10^{-4}	3.90×10^{-4}	1.24×10^{-3}	5.12×10^{-4}	5.61×10^{-4}	1.28×10^{-3}
0.4	s.s.	5.14×10^{-3}	2.66×10^{-3}	5.7×10^{-3}	2.9×10^{-3}	9.24×10^{-4}	2.02×10^{-3}
	c.c.	1.48×10^{-3}	7.67×10^{-4}	2.2×10^{-2}	1.11×10^{-2}	1.19×10^{-3}	2.62×10^{-3}
	c.f.	3.83×10^{-4}	1.99×10^{-4}	6.69×10^{-4}	1.74×10^{-4}	5.4×10^{-4}	1.21×10^{-3}

[a] $\omega_0^2 I_2/S_2 = 7.74 \times 10^{-4}$, and $\omega_0^2 I_3/S_3 = 1.72 \times 10^{-6}$.
[b] s.s. = simply supported, c.c. = clamped-clamped, and c.f. = clamped-free.

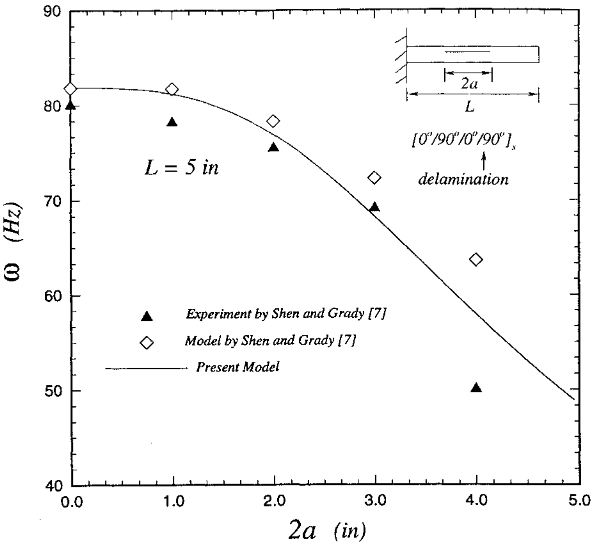

Fig. 3 Comparison of delamination length effect on natural frequency.

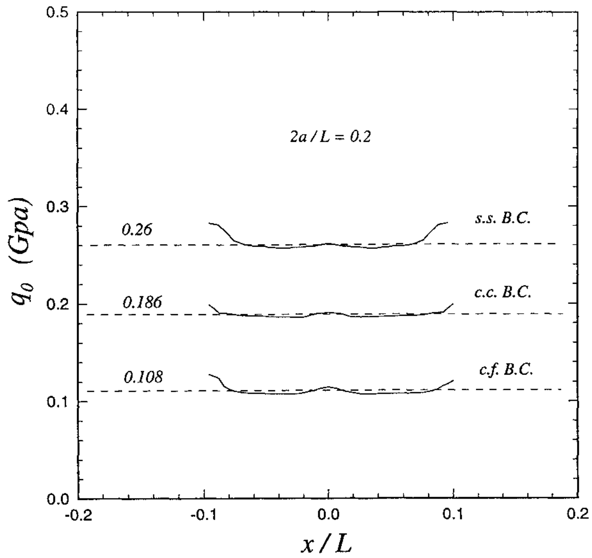

Fig. 4 Normal pressure distribution along delamination region ([$0_2/90_2/0_2$/honeycomb]$_s$).

$E_{11} = 105$ GPa, $E_{22} = 8.74$ GPa, $G_{12} = 4.56$ GPa, $\nu_{12} = 0.327$, $\rho_f = 1.6 \times 10^3$ kg/m^3, whereas those of the aluminum honeycomb core are $G_{xz} = 103$ MPa, $G_{yz} = 62.1$ MPa, and $\rho_c = 16$ kg/m^3.[14] The ply thickness t_{ply} is 0.125 mm, the length of the sandwich beam L is 100 mm, and the reference core thickness c_0 is equal to 10 mm. The sandwich construction [$0_2/90_2/0_2$/honeycomb]$_s$ is used as an example to study the effects of delamination and core on free vibration, whereas [θ_n/90/honeycomb]$_s$ is used to study the effect of fiber orientation. To study the effect of stacking sequence, we pile up the faces of the sandwiches by the laminate with fiber directions in 0 and ±45 deg. The reference natural frequency ω_0 used in the following figures and tables are then calculated from the perfect cantilever sandwich beams of [$0_2/90_2/0_2$/honeycomb]$_s$, [0_4/90/honeycomb]$_s$, and [0/−45/45/0/honeycomb]$_s$, respectively. The values are $\omega_0 = 295.54$ rad/s for the study of delamination and core effect, $\omega_0 = 307.49$ rad/s for the study of fiber orientation effect, $\omega_0 = 285.82$ rad/s and for the study of stacking sequence effect.

Before studying the effect of delamination, core, and face on the free vibration, we first check all of the assumptions stated in the previous sections: 1) normal pressure $q(= q_0 \sin \omega t)$ is uniform along the contact region, 2) P_i/S_i and $\omega_0^2 I_i/S_i$, $i = 2, 3$, are far smaller than unity, 3) γ_{xz_2} and γ_{xz_3} are far smaller than $\partial w_2/\partial x$, and 4) $2aq_i/S_i$, $i = 2, 3$, are far smaller than γ_{xz_2} and γ_{xz_3}. The numerical results given in Table 1 and Fig. 4 show that these assumptions are valid under small flexural vibration, usually $w/h \leq 0.1$.

A. Delamination Effect

The delamination lies symmetrically with respect to the midpoint of the beam. Figure 5 shows the relation between the first mode natural frequency ω/ω_0 and delamination length $2a/L$ for clamped-free ends as the core thickness c is reduced to be zero gradually. From this figure, it is shown that the existence of delamination will lower

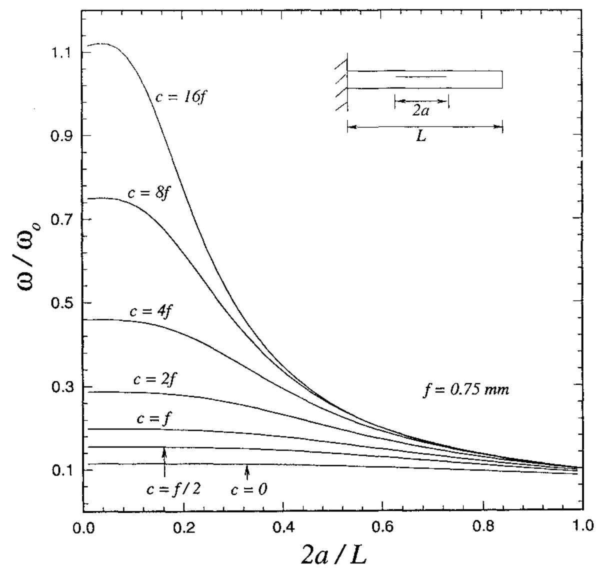

Fig. 5 Effect of delamination length on first mode natural frequency of cantilever composite sandwich beam ([$0_2/90_2/0_2$/honeycomb]$_s$, carbon/epoxy laminated faces).

the natural frequencies. The upper bound of natural frequencies denotes that of the perfect sandwich, whereas the lower bound denotes that of detached sandwich beam. Moreover, this figure reveals that the natural frequency of a delaminated composite sandwich beam decreases gradually when the core thickness decreases and will approach that of a delaminated composite laminate as the core thickness is reduced to zero. The results of delaminated composite

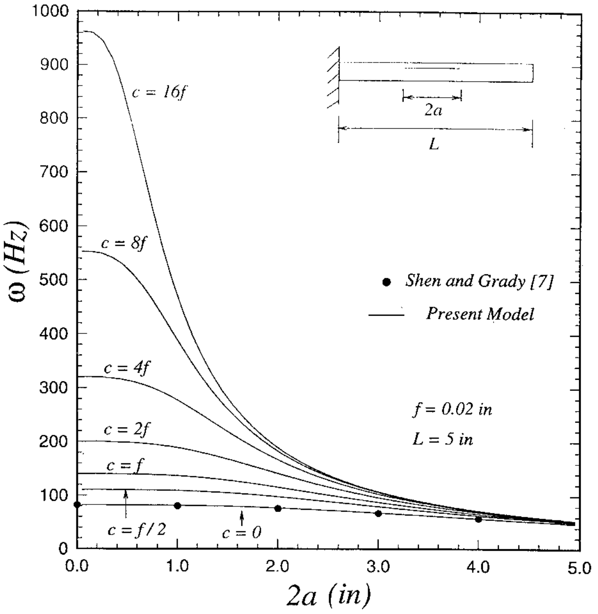

Fig. 6 Effect of delamination length on first mode natural frequency of cantilever composite sandwich beam ([0/90/0/90/honeycomb]$_s$, graphite/epoxy laminated faces).

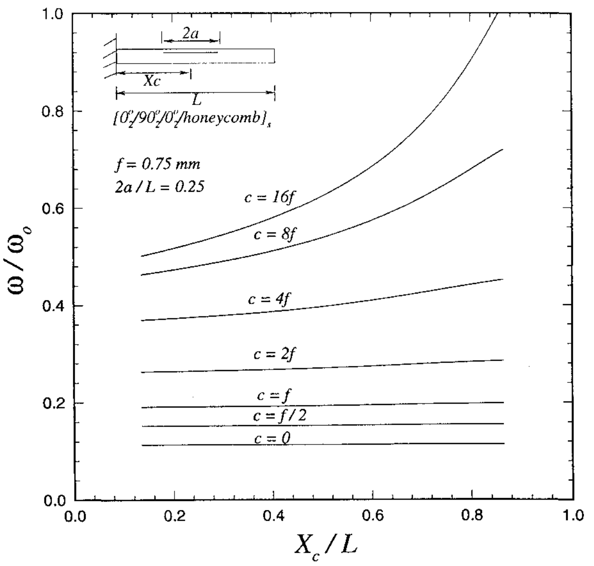

Fig. 7 Effect of spanwise location of delamination on first mode natural frequency of cantilever composite sandwich beam.

beams are represented by setting $c = 0$, which have been checked in the last section through the presentation of Fig. 3. To see more clearly, a counterpart of Fig. 5 with the face composed of the same composite laminates used by Shen and Grady[7] is plotted in Fig. 6. These results also mean that the model of delaminated composite sandwich proposed here is reliable.

The effects of delamination along spanwise locations on natural frequency for cantilever beam are shown in Fig. 7. It shows that the natural frequency of a delaminated composite sandwich will also approach that of a delaminated composite laminate when the core thickness is reduced to zero. The weakening effect of the delamination on the first mode natural frequency appears to become a minimum when the delamination is placed near the free end of a cantilever beam; that is, it appears on the high curvature of the first mode shape. As to the case of simply supported ends, the minimum weakening effect of the delamination occurs when the delamination is located symmetrically with respect to the midpoint of the beam.[18] Similar phenomena occur for delaminated composite beams, which have been explained by Tracy and Pardoen,[6] that the effect of the delamination is reduced as the delamination moves from regions of high shear force to regions of high curvature. The same conclusions are also proposed by Mujumdar and Suryanarayan[5] for delaminated isotropic beams.

As for free vibration mode shapes, Fig. 8 shows that the mode shapes of a delaminated composite sandwich beam will approach those of a delaminated composite laminate as the core thickness is reduced to zero. It also shows that the mode shapes of delaminated sandwich beams differ from those of perfect sandwich beams, especially in the region of delamination because the effective bending stiffnesses D_i, $i = 2, 3$, of the delaminated region is smaller than that of region 1. Therefore, the deflection pattern is significantly influenced by the delamination.

B. Core Effect

The effect of core on the natural frequency is usually discussed by considering the thickness and effective transverse shear modulus separately, since the transverse shear stiffnesses S_i, $i = 1, 2, 4$, given in Eq. (2d), are related to core thickness c and transverse shear modulus G_{xz} of the core. Figure 9 shows the relation between natural frequency and transverse shear modulus G_{xz} for various central delamination sizes for clamped-free boundary conditions, where $G_{xz0} = 0.103$ GPa, and the core thickness c_0 is kept as 10 mm. The figure shows that the natural frequency depends significantly on the transverse shear modulus when the delamination length is short, whereas there is almost no influence on frequency for longer delaminations. This phenomenon may be explained by thinking that the contribution of the core is relatively small in the delamination region. Therefore, the longer the delamination, the smaller the transverse shear modulus effect.

To study the effect of core thickness on the natural frequency, the transverse shear modulus G_{xz} is kept fixed and set to be equal to

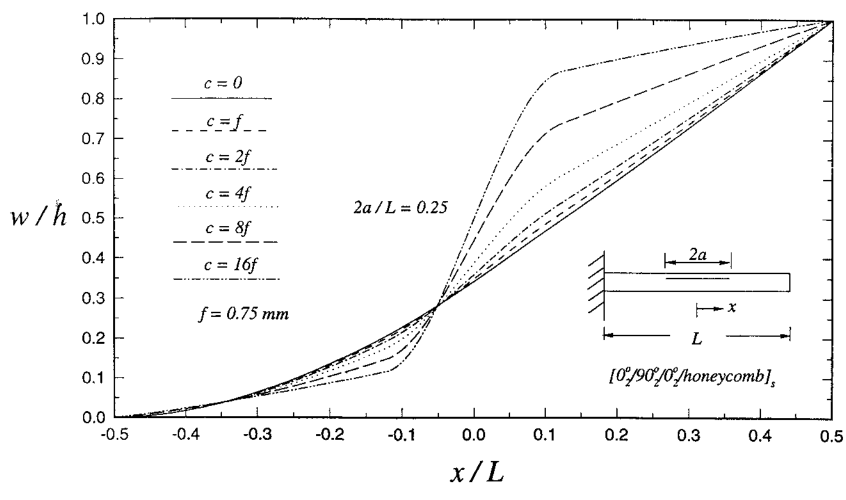

Fig. 8 Effect of delamination length on first mode shape of cantilever sandwich beam.

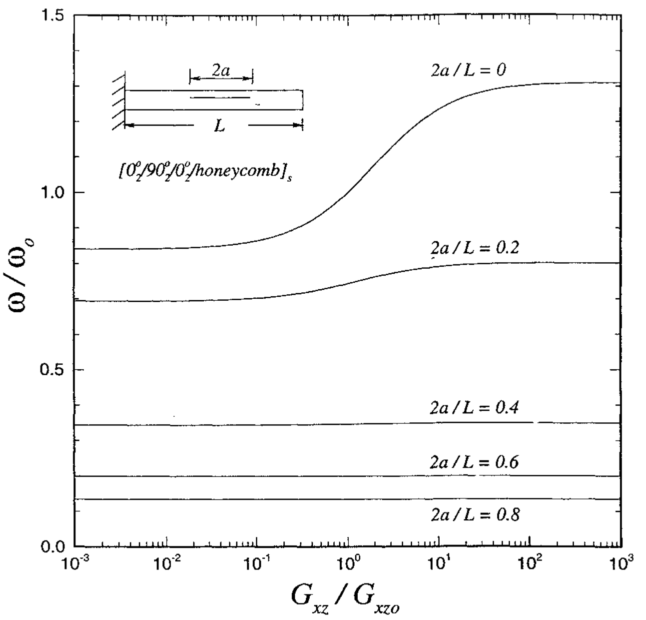

Fig. 9 Effect of transverse shear modulus on first mode natural frequency of cantilever delaminated composite sandwich beam.

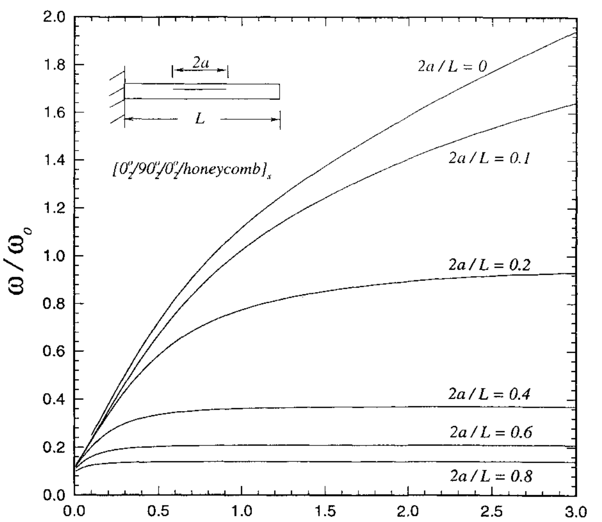

Fig. 10 Effect of core thickness on first mode natural frequency of cantilever delaminated composite sandwich beam.

G_{xz_0}. The results are shown in Fig. 10 for various central delamination length. As in the case of the transverse shear modulus, the natural frequency depends on the core thickness significantly when the delamination is short, whereas there is almost no influence on frequency for longer delaminations.

C. Face Effect

To study the effect of composite sandwich face, we consider the thickness, fiber direction, and stacking sequence of the laminate composite. Before the calculation, we may expect that the natural frequency will increase when the face thickness increases, or when the fiber is oriented to the direction of x axis shown in Fig. 1. Figure 11 verifies this expectation. A series of composite sandwich construction $[\theta_n/90/\text{honeycomb}]_s$ are used in this case. The reference parameters are $f_0 = 0.625$ mm (i.e., $n = 4$), $\theta_0 = 90$ deg, and $2a/L = 0.2$. The thickness effect is considered by varying the number of $\theta = 0$ deg lamina, i.e., changing n from 0 to 4.

As for the stacking sequence, we consider five different sequences shown in Table 2. For different delamination length under clamped-free boundary conditions, this table shows that the natural frequency of the delaminated sandwich with stacking sequence $ss1$ is always

Table 2 Effect of stacking sequence on the natural frequency of delaminated sandwich [composite laminate/honeycomb]$_s$ beams with clamped-free ends

	ω/ω_0				
$2a/L$	$ss1^a$	$ss2^a$	$ss3^a$	$ss4^a$	$ss5^a$
0.2	0.571	0.492	0.459	0.457	0.409
0.3	0.348	0.292	0.269	0.269	0.237
0.4	0.236	0.196	0.180	0.180	0.158
0.5	0.173	0.143	0.131	0.131	0.115

[a] $ss1 \equiv [0/-45/45/0]$, $ss2 \equiv [-45/0/45/0]$, $ss3 \equiv [0/0/-45/45]$, $ss4 \equiv [45/-45/0/0]$, and $ss5 \equiv [-45/0/0/45]$.

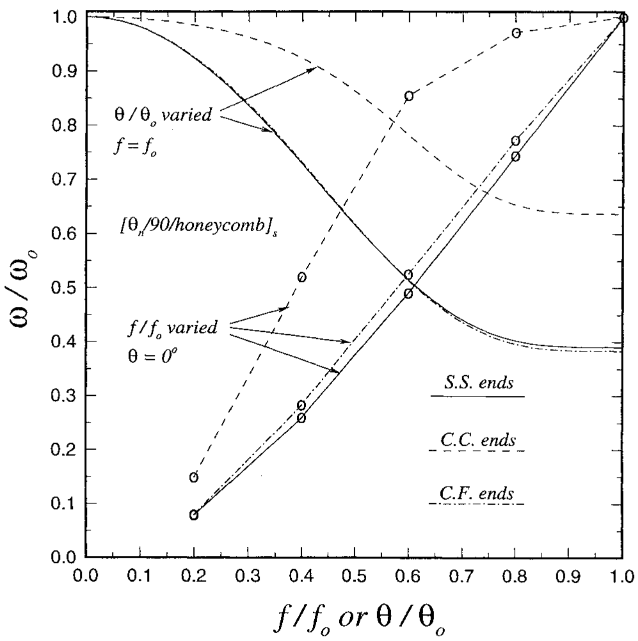

Fig. 11 Effect of face on first mode natural frequency of delaminated composite sandwich beam.

the highest one, whereas that of $ss5$ is always the lowest one. The same conclusions are also found for simply supported and clamped-clamped boundary conditions.[18] The reasons are the same as those described in Hwu and Hu's paper[15] for the effect of stacking sequence on buckling load; i.e., the natural frequency depends on the effective bending stiffnesses D_2 and D_3 of the delaminated regions, not D_1. This means that the order of the relative natural frequencies shown in this table is $ss1 > ss2 > ss3 > ss4 > ss5$, which is consistent with the order of D_3(or D_2), not D_1.

VI. Conclusion

A one-dimensional model of free vibration behavior of delaminated composite sandwich beams considering transverse shear effect and rotary inertia has been established in this paper. The solutions are general and can be reduced to solve the free vibration problem of delaminated composite beams and perfect sandwich beams. When the core thickness is close to zero, the solutions of the present model approach those of delaminated composite beam, which verifies the present model. Meanwhile, the effect of core, faces, and delamination on natural frequencies and the associate mode shapes are also discussed in this paper.

Acknowledgment

The authors would like to thank the National Science Council, Republic of China, for support through Grant NSC 82-0401-E006-356.

References

[1]Kulkarni, S. V., and Frederick, D., "Frequency as a Parameter in Delamination Problems—A Preliminary Investigation," *Journal of Composite Materials*, Vol. 5, Jan. 1971, pp. 112–119.

[2]Ramkumar, R. L., Kulkarni, S. V., and Pipes, R. B., "Free Vibration Frequencies of a Delaminated Beam," *34th Annual Technical Conference Proceedings*, Reinforced/Composite Inst., Society of Plastics Industry, Inc., Sec. 22-E, 1979, pp. 1–5.

[3]Cawley, P., and Adams, R. D., "A Vibration Technique for Non-Destructive Testing of Fiber Composite Structures," *Journal of Composite Materials*, Vol. 13, April 1979, pp. 161–175.

[4]Wang, J. T. S., Liu, Y. Y., and Gibby, J. A., "Vibrations of Split Beams," *Journal of Sound and Vibration*, Vol. 84, No. 4, 1982, pp. 491–502.

[5]Mujumdar, P. M., and Suryanarayan, S., "Flexural Vibrations of Beams with Delaminations," *Journal of Sound and Vibration*, Vol. 125, No. 3, 1988, pp. 441–461.

[6]Tracy, J. J., and Pardoen, G. C., "Effect of Delamination on the Natural Frequencies of Composite Laminates," *Journal of Composite Materials*, Vol. 23, No. 12, 1989, pp. 1200–1215.

[7]Shen, M. H., and Grady, J. E., "Free Vibration of Delaminated Beams," *AIAA Journal*, Vol. 30, No. 5, 1992, pp. 1361–1370.

[8]Yu, Y. Y., "A New Theory of Elastic Sandwich Plates—One-Dimensional Case," *Journal of Applied Mechanics*, Vol. 26, No. 3, 1959, pp. 415–421.

[9]Yu, Y. Y., "Simplified Vibration Analysis of Elastic Sandwich Plates," *Journal of the Aerospace Sciences*, Vol. 27, No. 12, 1960, pp. 894–900.

[10]Yu, Y. Y., "Flexural Vibrations of Elastic Sandwich Plates," *Journal of the Aerospace Sciences*, Vol. 27, No. 4, 1960, pp. 272–282.

[11]Monforton, G. R., and Ibrahim, I. M., "Analysis of Sandwich Plates with Unbalanced Cross-Ply Faces," *International Journal of Mechanical Sciences*, Vol. 17, No. 3, 1975, pp. 227–238.

[12]Monforton, G. R., and Ibrahim, I. M., "Modified Stiffness Formulation of Unbalanced Anisotropic Sandwich Plates," *International Journal of Mechanical Sciences*, Vol. 19, No. 6, 1977, pp. 335–343.

[13]Ibrahim, I. M., Farah, A., and Rizk, M. N. F., "Dynamic Analysis of Unbalanced Anisotropic Sandwich Plates," *Journal of Engineering Mechanics*, Vol. 107, No. 2, 1981, pp. 404–418.

[14]Kanematsu, H. H., and Hirano, Y., "Bending and Vibration of CFRP—Faced Rectangular Sandwich Plates," *Composite Structure*, Vol. 10, No. 2, 1988, pp. 145–163.

[15]Hwu, C., and Hu, J. S., "Buckling and Postbuckling of Delaminated Composite Sandwich Beam," *AIAA Journal*, Vol. 30, No. 7, 1992, pp. 1901–1909.

[16]Vinson, J. R., and Sierakowski, R. L., *The Behavior of Structures Composed of Composite Materials*, Martinus–Nijhoff, Dordrecht, The Netherlands, 1986.

[17]Raville, M. E., Ueng, E., and Lei, M., "Natural Frequencies of Vibration of Fixed-Fixed Sandwich Beams," *Journal of Applied Mechanics*, Vol. 28, No. 3, 1961, pp. 367–371.

[18]Hu, J. S., "Structural Analysis of Delaminated Composite Sandwich Beams," Ph.D. Dissertation, Inst. of Aeronautics and Astronautics, National Cheng Kung Univ., Taiwan, ROC, 1994.